Differential Calculus with Rational Coefficients

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The purpose of this note is to explain how to perform differential calculus on a general simplicial complex and to extract homotopy information beyond de Rham cohomology. We shall see that the infinite part of homotopy groups (and more) becomes readily computable from this general differential calculus. Applications to geometry (e.g. Kähler manifold, complex variety, loop space, classification of manifolds) will be emphasized.

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Chapter 1

Vector fields and differential forms

Differential calculus is usually carried out on smooth spaces, involving two models of derivatives–vector fields and differential forms. The former is geometric, covariant and easier for visualization; the latter is algebraic, contravariant and easier for calculation. We ask here

- Can we extend differential calculus to non-smooth spaces, such as pseudo-manifolds, topological manifolds, simplicial complex or even more complicated spaces like those arise from dynamic systems (e.g. Mandelbrot set)? Can we define vector fields? Can we define differential forms? A reasonable definition should be compatible with known results.

- Can we use a generalized differential calculus to obtain well-known theorems about vector fields and differential forms for a general space? To name a few, Poincaré-Hopf index theorem, de Rham isomorphism, Poincaré-Lefschetz-Alexander duality. If we have some sort of geometry can we get a well-defined Hodge theory, Chern-Weil theory?

- Can we obtain more information beyond the classical theorems? What is the limit of the information we can get?

- What more can we say about manifolds by generalizing differential calculus to a larger class of spaces? Such as distinguishing manifolds from other spaces.

1.1   Cartan-Chevalley-Eilenberg-Koszul duality

We discuss here a general relation between the two models of derivatives–vector fields and differential forms; it is clear they are in some sort of duality, we shall make it clear. For simplicity, we restrict ourselves to a finite dimensional situation by working with left invariants objects on Lie groups.

Let G be a compact connected Lie group, let us compute its de Rham cohomology. Choose a left invariant measure on G, which exists by averaging an arbitrary measure $d\mu$ via left translation. Now wedge products of left invariant forms are left invariant, and the exterior d preserves left invariance of differential forms; therefore left invariant forms, among all forms, is a differential graded subalgebra. Then, averaging over the group yields a splitting commuting with d from all forms into left invariant forms. This implies the cohomology of left invariant forms embeds into the de Rham cohomology of G. On the other hand, since G is connected, pulling back by left translation is homotopy equivalent to identity and therefore acts trivially on cohomology. Therefore every class in de Rham cohomology can in fact be represented by a left invariant form. So, we can compute the de Rham cohomology of G using left invariant forms which is finite dimensional.

Now observe that left invariant forms are dual to left invariant (poly) vector fields, the left invariant de Rham complex is given by

$$(\wedge^\bullet \mathfrak{g}^\vee, d)$$

where $\mathfrak{g}$ is the Lie algebra of G identified with left invariant vector fields on G. The exterior d is identified as follows.

The Lie bracket on $\mathfrak{g}$ is a skew-symmetric map

$$[-,-] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}.$$

We obtain a map by dualizing the above

$$\mathfrak{g}^\vee \to \mathfrak{g}^\vee \wedge \mathfrak{g}^\vee$$

The exterior differential d is a derivation extended from this map (via Leibniz rule) on the free graded commutative algebra $\wedge^\bullet \mathfrak{g}^\vee$. One verifies that $d \circ d = 0$ is equivalent to Jacobi identity.

To summarize

Theorem 1.1 (Cartan-Chevalley-Eilenberg). *Let G be a compact connected Lie group and $\mathfrak{g}$ its Lie algebra. Let $\wedge^\bullet \mathfrak{g}^\vee$ be the free graded commutative algebra generated by the dual of $\mathfrak{g}$, and let d be the derivation on $\wedge^\bullet \mathfrak{g}^\vee$ extended from the map dual to the structural map of $\mathfrak{g}$*

$$[-,-] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}.$$

Then the cohomology ring of $(\wedge^\bullet \mathfrak{g}, d)$ is isomorphic to the de Rham cohomology ring of G, i.e.

$$H^*(\wedge^\bullet \mathfrak{g}^\vee, d) \cong H^*_{dR}(G).$$

This theorem reveals a fundamental relation between Lie algebras and (commutative) differential graded algebras. This relation turns out to be so general that continues to hold much beyond the context of Lie groups. We will refer to differential graded algebras of the form $(\wedge^\bullet \mathfrak{g}^\vee, d)$ as *dual Lie algebras*.

Exercise 1.2. Calculate the de Rham cohomology of $U(1)$, $Sp(1)$, $SU(3)$, $Spin(7)$ and G_2.

Notice that the theorem in fact holds for non-compact groups whose Lie algebras are not necessarily semi-simple. In general $\mathfrak{g}$ has the structure of an extension of a semi-simple Lie algebra by a solvable ideal

$$0 \to \mathrm{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}_{s.s.} \to 0.$$

This in turn implies that the dual Lie algebra $\wedge^\bullet \mathfrak{g}$ can be formed in the following two steps:

(i) Form the dual Lie algebra B of a semi-simple Lie algebra (a direct sum of simple Lie algebras), which is a tensor product of simple dual Lie algebras. Note that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ for $\mathfrak{g}$ semi-simple, hence the differential d on its dual Lie algebra is injective in degree one. Let $\{x_k\}$ denote a dual basis for $\mathfrak{g}$, then the differential has the general form

$$dx_k = \sum a_{ij}^k x_i \wedge x_j$$

(ii) Then form an extension $A = B(x_\alpha, x_\beta, ...)$ by adding in new generators $\{x_\alpha\}$ dual to a basis of the radical ideal with differential of the form

$$dx_\alpha = \sum \theta_{\alpha\beta} x_\beta + a_\alpha$$

where $\theta_{\alpha\beta}$ and a_α are taken from B in degree one and two respectively. Notice that $d \circ d = 0$ implies that $\Theta = (\theta_{\alpha\beta})$ and $a = (a_\alpha)$ satisfy

$$d\Theta - \Theta \circ \Theta = 0, \quad d_\Theta a = 0$$

where $d_\Theta = d - \Theta$.

1.2 Vector fields on singular spaces

Let X be a space divided into (disjoint) pieces D_i (strata) on which vector fields are defined and assume such division is nice enough to allow gluing. Examples of such spaces include complex and real algebraic or analytic varieties and simplicial complexes. In these examples, the space is divided into smooth manifolds of various dimensions in such a way that if a sequence of points on

a larger dimensional piece (higher stratum) converge to a point in a lower dimensional one (lower stratum), then the corresponding tangent spaces converge to a vector space containing the lower dimensional tangent space. Therefore one can talk about continuous vector fields.

Theorem 1.3 (M. H. Schwartz)**.** *Let V be a radial vector field on X with isolated singularity, then the Euler characteristic of X is equal to the index of V.*

A *radial* vector field is one which points normal towards the higher stratum near the place where a lower stratum is attached to a higher stratum. A vector field already defined on a lower stratum can be radially extended by gradually adding a vector field normal towards the higher stratum. For example, a vector field defined on the boundary of a manifold can be extended to a collar neighborhood by gradually adding an inward normal vector field.

The radial condition is necessary to make sense of index. Suppose a vector field has an isolated singularity in a lower stratum, which is contained in the closure of some higher stratum. Then one can compute the index of the vector field restricted to the lower stratum or compute in the higher stratum, the results may differ if the vector field is not radial.

Proof. This goes by direct induction on strata using Poincaré-Hopf theorem for manifold with boundary. ∎

Proposition 1.4 (Borel, Deligne, Sullivan)**.** *The Euler characteristic of the link to a point in a complex algebraic variety is zero.*

Sketch of an unpublished proof due to Borel. Choose a radial vector field near the point, apply J, the almost complex structure, to the vector field and then project to the link to obtain a radial vector field on the link without singularity. ∎

Corollary 1.5. *The Euler characteristic of the link to a point in a real algebraic variety is zero modulo* 2*.*

Proof. Complexify the real variety. ∎

Question 1.6*.* Can we define Lie bracket for vector fields on a singular space? If not, what about Lie infinity structure?

1.3   Differential forms on simplicial complexes

We analyze differential forms on simplicial complexes and prove a corresponding de Rham theorem.

1.3.1 Definition of differential forms

Let X be a simplicial complex, which is a union of a collection (assume finite for our discussion) of standard simplicies of the form

$$\Delta^n = \{t_0 + t_1 + \cdots + t_n = 1, t_i \geq 0\} \subset \mathbb{R}^{n+1}.$$

These simplicies should be glued together in such a nice way that the intersection of any two simplicies, if non-empty, should also be sub-simplex of each.

By a differential k-form on the standard simplex Δ^n, we mean an expression of the form

$$\sum_{I=(i_1 < \cdots < i_k)} \phi_I(t_0, \ldots, t_n) dt_{i_1} dt_{i_2} \ldots dt_{i_k}.$$

Depending on the choice of coefficient functions ϕ, we obtain various notions of differential forms, such as

(i) smooth functions, analytic functions, C^1-functions, Lipschitz functions, polynomial functions etc.[1]

(ii) polynomial functions with coefficients in a field $\mathbb{F}$ (or a ring); here $\mathbb{F}$ can be any sub-field of $\mathbb{C}$, or not e.g. $\mathbb{F}_p$.

With either fixed choice, we obtain a differential graded algebra of forms on Δ^n, denoted by $\mathscr{A}(\Delta^n)$. If we want to emphasize the choice made, we can write $\mathscr{A}_{C^\infty}, \mathscr{A}_{\mathbb{Q}-poly.}$ etc.

Now by a differential forms on X we mean a collection of differential forms ω_σ on each simplex σ of X satisfying a natural gluing condition, that if $i : \tau \to \sigma$ is an inclusion of a face, then $i^* \omega_\sigma = \omega_\tau$. Then since restriction (or in general pull-back) commutes with wedge product and exterior d, we obtain a differential graded algebra of forms $\mathscr{A}(X)$ on X. Let us call the cohomology of $\mathscr{A}(X)$, the *de Rham cohomology* of X. Again if we want to emphasize our choice of the coefficients, we can add adjectives.

1.3.2 De Rham theorem

It is natural to ask whether the de Rham cohomology coincides with singular cohomology of X, and a version of de Rham theorem is anticipated. For this, we first point out that there is a natural

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[1] it is a subtle point to make exterior d well-defined; one choice is to require the coefficients to be 1-differentiable, but this is not necessary because exterior d only captures an "average" of derivatives, see Whitney [**whitney**] for a general treatment.

integration map, first defined on each simplex by the standard integration[2] and extended to all forms by integration simplex-by-simplex

$$\mathscr{A}(X) \xrightarrow{\int} C^{\bullet}(X; \mathbb{F}).$$

Here $C^{\bullet}(X; \mathbb{F})$ means simplicial cochain complex with $\mathbb{F}$-coefficients, the field $\mathbb{F}$ depends on our notion of forms.

It is clear that the integration map is a map of cochain complexes (with respect to d and coboundary), and therefore induces a (graded) map

$$H_{dR}(X) \to H^{\bullet}(X; \mathbb{F}).$$

We prove this map is an additive isomorphism for characteristic zero field $\mathbb{F}$, by applying induction on dimension of cells.

Let X_k denote the union of all simplicies in X up to dimension k, namely the k-skeleton of X. We attempt to apply induction to prove the anticipated de Rham theorem by induction and thus assuming the isomorphism holds for X_{k-1}. Then we are reduced to considering the triple

$$X_{k-1}, X_k, (X_k, X_{k-1})$$

Assume we have a long exact sequence

$$\cdots \to H_{dR}(X_k, X_{k-1}) \to H_{dR}(X_k) \to H_{dR}(X_{k-1}) \to \cdots$$

Then by five lemma and induction hypothesis, we are reduced to prove the de Rham theorem form the pair (X_k, X_{k-1}). Here by forms on the pair (X_k, X_{k-1}) we mean forms on X_k that vanishes on X_{k-1}. Notice that forms on the pair (X_k, X_{k-1}) is nothing but a collection of forms on k-simplicies in X that vanish on their boundaries. Therefore, we are reduced to proving de Rham theorem for the pair $(\Delta^k, \partial \Delta^k)$. Then again apply the above consideration to the triple $\partial \Delta^k, \Delta^k, (\Delta^k, \partial \Delta^k)$ it suffices to prove de Rham theorem for the simplex Δ^k; this means we need to prove a Poincaré lemma.

On the other hand, the long exact sequence needed in the above argument will follow from the following short exact sequence of cochain complexes by standard homological algebra.

$$0 \to \mathscr{A}(X_k, X_{k-1}) \to \mathscr{A}(X_k) \to \mathscr{A}(X_{k-1}) \to 0.$$

The first map is injective by definition: $\mathscr{A}(X_k, X_{k-1})$ is exactly the kernel of the second restriction map. However the surjectivity of the second map needs to be verified, that is we need to show that

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[2] caution: I do not know yet how to integrate with positive characteristic coefficients.

every form on X_{k-1} can be extended to a form on X_k; this can be reduced to showing every form on $\partial \Delta^k$ can be extended to Δ^k.

In summary, we have the following (conditional) de Rham theorem.

Theorem 1.7 (de Rham isomorphism)**.** *The integration map induces an isomorphism on cohomology, provided the following two properties holds.*

(i) *(Local acyclicity, i.e. Poincaré lemma) $H^i_{dR}(\Delta^n) = 0$ for $i > 0$. That is, every (positive degree) closed form on Δ^n is exact.*

(ii) *(Flasqueness, i.e. Extension lemma) Every form on $\partial \Delta^n$ can be extended to a form on Δ^n.*

Remark 1.8. The definition of differential forms can be extended to a general space as long as it is obtained by gluing in a nice way some notion of *cells* on which differential forms can be defined. And the de Rham theorem will hold if *boundary* makes sense and the above two properties hold: differential forms give the correct cohomology of the cells and one can always extend forms from boundary to cells. Note further that differential forms can be replaced by other algebraic objects; of course the integration map has to be modified accordingly.

Remark 1.9. One might wonder why we need one more algebraic object associated to a simplicial complex that computes the singular cohomology, especially when it is simple enough to compute (using computer for instance) singular cohomology using simplicial cochains. The benifit is that the algebra of differential forms carries a (graded) commutative product on the nose from which it is easier to extract further invariants of the space beyond cohomology. From modern point of view, all the homotopy information is contained in singular cochains, but much more complicated structures should be taken into consideration which makes computation almost impossible; by contrast computation with differential graded algebras is easy. Moreover, differential forms are clearly closely related to geometry.

We now proceed to prove the two lemmas needed.

1.3.3 Poincaré lemma

We now prove

Lemma 1.10 (Poincaré lemma)**.** *Closed forms on Δ^n are exact, provided characteristic zero coefficients are used.*

Proof. The intuition, of course, is to use that the standard simplex deformation retracts to a point; so we strengthen the statement to the following: let $c(K)$ be the cone over a simplicial complex K

(a general point is of the form $s \cdot k + (1-s) \cdot c$ where $k \in K$ and c is the cone point), then every closed form on $c(K)$ is exact. Since the cone over a standard simplex is again a standard simplex (of one higher dimension), the cone over a simplicial complex is again a simplicial complex.

Consider the deformation retraction of linear speed:

$$\mu : c(K) \times [0,1] \to c(K), \quad s \cdot k + (1-s) \cdot c \mapsto s(1-t) \cdot k + (1-s+st) \cdot c$$

Notice that pulling back by μ preserves all notions of differential forms under consideration.

Now we claim that if ω is a closed form on $c(K)$ then

$$d \left(\int_{t=0}^{t=1} \mu^* \omega \right) = \omega.$$

To see this, separate the variables from $c(K)$ and $[0,1]$ and write

$$\mu^* \omega = \alpha(t) + \beta(t) dt$$

where $\alpha(t), \beta(t)$ are t-parametrized forms on $c(K)$ (see remark below), subject to conditions:

(i) $\alpha(0) = \omega$, because the restriction of μ to $t = 0$ is identity.

(ii) $\alpha(1) = 0$, because the restriction of μ to $t = 1$ is the constant map (to the cone point).

(iii) $d\alpha(t) + d\beta(t) \cdot dt = 0$, because $d\omega = 0$ and thus $d(\mu^*\omega = 0)$.

Then

$$d \int_{[0,1]} \mu^* \omega = d \int_{[0,1]} \beta(t) dt = \int_{[0,1]} d\beta(t) dt = \int_{[0,1]} -d\alpha(t) = \alpha(0) - \alpha(1) = \omega.$$

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Remark 1.11. The above argument breaks with positive characteristic coefficients. If we write $\beta(t) = \sum_{i \geq 0} \beta_i t^i$ for $\beta_i \in \mathscr{A}(c(K))$, then the formula $\int_{[0,1]} \beta(t) dt = \sum_{i \geq 0} \beta_i / (i+1)$ contains all positive integers as denominators. In fact, the de Rham cohomology for Δ^1, calculated using polynomial functions with finite coefficients is infinite dimensional.

1.3.4 Extension lemma

Now we prove the extension lemma and thus complete the proof of our de Rham theorem.

Lemma 1.12. *Every form on $\partial \Delta^n$ can be extended to a form on Δ^n; provided we use smooth functions or polynomial functions in our definition of differential forms.*

Proof. This is standard for smooth forms; we prove for polynomial forms using ideas from projective geometry. One can see the following argument easily fails for analytic forms.

Let us start by extending a polynomial form defined on one face, say $\{t_n = 0\}$, to the whole simplex Δ^n. For this, consider the stereographic projection onto the face $\{t_n = 0\}$

$$\pi : (t_0, t_1, \ldots, t_n) \mapsto \left(\frac{t_0}{1-t_n}, \frac{t_1}{1-t_n}, \ldots, \frac{t_{n-1}}{1-t_n}\right)$$

which is well-defined except for the vertex $\{t_n = 1\}$. We aim to extend the pull-back $\pi^*\omega$ of a polynomial form ω defined on $\{t_n = 0\}$ over to the vertex $\{t_n = 1\}$. Notice that ω is a polynomial in terms of $t_0, \ldots, t_{n-1}, dt_0, \ldots, dt_{n-1}$; therefore $\pi^*\omega$ is a polynomial in terms of $\frac{t_0}{1-t_n}, \ldots, \frac{t_{n-1}}{1-t_n}$ and $d(\frac{t_k}{1-t_n}) = \frac{dt_k}{1-t_n} + \frac{t_k}{(1-t_n)^2}$. So, $\pi^*\omega$ carries a removable singularity at the vertex $\{t_n = 1\}$, that is $(1-t_n)^N \pi^*\omega$ extends over to the vertex for N sufficiently large; but the restriction of $(1-t_n)^N \pi^*\omega$ to $\{t_n = 0\}$ is still ω; this is the extension we desire. Note that this way of extension has the property that if ω vanishes on the some part of the boundary of the face, then the extension vanishes on its cone (with respect to the vertex in consideration).

Next, to extend a given form defined on $\partial\Delta^n$, we first extend the restriction of this given form on one face as above to get a form ω_1 and then we extend the difference between ω_1 and the restriction of the given form on a second face to get a form ω_2; proceed further to extend the difference between $\omega_1 + \omega_2$ and the restriction of the given form on a third face so on and so forth. We end up with a form $\omega_1 + \cdots + \omega_n$ on Δ^n extending the given form on $\partial\Delta^n$. ∎

Question 1.13. Is there an example of an analytic form which does not extend at all?

1.3.5 Multiplicative structure

The usual de Rham isomorphism for smooth manifolds is further an algebra isomorphism with respect to wedge product of forms and cup product on cohomology. The same holds in our situation due to the local nature of the wedge product; see Whitney.

Recall that the cup product in singular cohomology can be defined using cross product and diagonal map; one forms the cross product of cocycles $\alpha \times \beta$ on $X \times X$ and then pull back the resulting cohomology class along the diagonal map $\Delta : X \to X \times X$. Similar holds for wedge product of differential forms; one forms the tensor product of forms on $X \times X$ and restrict to diagonal to obtain wedge product.

The only catch is that $X \times X$ does not inherit a canonical simplicial complex structure from that of X (e.g. $\Delta^1 \times \Delta^1$). However, $X \times X$ carries a natural *linear cell complex* structure[3], which can

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[3]a linear cell complex is a union of convex linear cells such that on the intersection of two cells the linear structures agree.

be refined into a simplicial complex via subdivision. And the tensor product of differential forms naturally becomes a differential form on any a subdivision of $X \times X$.

Notice that if we are dealing with rational polynomial forms, then we should further require the subdivision to be rational in the sense that each new vertex has rational barycentric coordinates in $X \times X$.

Proposition 1.14. *Let X be a linear cell complex and X' a subdivision of X into a simplicial complex. Then the integration map*

$$\mathscr{A}(X') \xrightarrow{\int} C^{\bullet}(X)$$

induces an isomorphism on cohomology. Here the integration map is defined by integrating forms on X' over cells of X which are unions of simplicies of X'.

Proof. Apply the de Rham theorem for each cell of X and argue by induction over the cells. ∎

As an easy corollary we have

Theorem 1.15. *The de Rham isomorphism is an algebra isomorphism.*

Proof. Exercise. ∎

1.3.6   Comparison to smooth de Rham isomorphism

For a smooth manifold equipped with a smooth triangulation, we have both smooth forms in the usual sense and (real or rational) polynomial forms we considered above. These two kinds of forms can be included into a bigger set of forms–piecewise smooth forms whose cohomology is also isomorphic to the singular cohomology via integration. The proof is left as an exercise.

Chapter 2

Homotopy theory of differential graded algebras

In this chapter, we study differential graded algebras (DGAs) in their own right; algebraic invariants of a DGA beyond its cohomology will be extracted. In the next chapter, we apply the results of this chapter to obtain invariants of a space beyond its singular cohomology.

2.1 Minimal model, first glance

DGAs obtained from topological spaces, either C^∞ forms on manifolds or polynomial forms discussed in the previous chapter, are often not finite dimensional, not even finitely generated as an algebra and even worse not of finite type (finitely many generators in each degree). It is a natural idea in algebra to find well-behaved presentations of an algebraic object in order for further understanding (e.g. surface groups); here it would be desirable to find for a DGA a replacement of some finite nature without loosing much information (e.g. preserving cohomology). One native way is to perform the following procedures.

Suppose we have a DGA $\mathscr{A}$ over $\mathbb{Q}$, assume it is connected in degree zero $\mathscr{A}^0 = \mathbb{Q}$ and simply-connected $H^1(\mathscr{A}) = 0$ for simplicity, we aim to build a new DGA $\mathscr{M} = \mathscr{M}(\mathscr{A})$ with a DGA map $\mathscr{M} \to \mathscr{A}$ preserving the cohomology of $\mathscr{A}$, i.e. it induces an isomorphism on cohomology; and we want to build $\mathscr{M}$ as small as possible; intuitively we should build $\mathscr{M}$ by adding generators into $\mathscr{M}$ to generate the cohomology of $\mathscr{A}$ and adding further generators into $\mathscr{M}$ to produce relations. More precisely, we inductively build a sequence of free DGAs $\mathscr{M}_k$ generated in degrees $\leq k$ together with maps $\mathscr{M}_k \to \mathscr{A}$ so that

(i) $H^i(\mathscr{M}_k) \to H^i(\mathscr{A})$ is an isomorphism for $i \leq k$ and injective for $i = k+1$; $\mathscr{M}_2 \to \mathscr{A}$ can be

built by choosing a set of representatives for a basis of $H^2(\mathscr{A})$, and $\mathscr{M}_2$ is the free algebra generated in degree 2 on the set of representatives with $d = 0$, together with the canonical map into $\mathscr{A}$; note that $H^3(\mathscr{M}_2) = 0$.

(ii) Inductively we add two sets of new generators of degree k to $\mathscr{M}_{k-1}$ to form $\mathscr{M}_k$; one set $\{x_\alpha\}$ with $dx_\alpha = 0$ to generate the rest of $H^k(\mathscr{A})$ not mapped onto by $\mathscr{M}_{k-1}$; then another set $\{y_\beta\}$ with dy_β killing the kernel of the map $H^{k+1}(\mathscr{M}_{k-1}(x_\alpha)) \to H^{k+1}(\mathscr{A})$. We note that since there is nothing in degree one, adding in x_α would not create new relations in degree k, so all relations to be killed in the second step arise from generators of $\mathscr{M}_{k-1}$. So

$$\mathscr{M}_k = \mathscr{M}_{k-1}(x_\alpha, y_\beta), \quad dx_\alpha = 0, dy_\beta = P_\beta(\cdots)$$

where $P_\beta' s$ are polynomials in generators of $\mathscr{M}_{k-1}$ (of degree at least two).

Example 2.1. Let $\mathscr{A}$ be the free algebra generated by x, y, z with $|x| = |y| = 3$ and $|z| = 5$; $dx = dy = 0, dz = xy$. Let $\mathscr{B}$ be the cohomology algebra of $\mathscr{A}$ with $d = 0$. Then $\mathscr{M}(\mathscr{A}) = \mathscr{A}$. We leave it as an exercise for the reader to find $\mathscr{M}(\mathscr{B})$ and show that it is not isomorphic to $\mathscr{A}$. (Hint: $\mathscr{M}(\mathscr{B})$ has non-trivial generators in degree 8.)

It is clear that $\mathscr{M} = \cup_k \mathscr{M}_k$ is a free algebra whose cohomology is isomorphic to $\mathscr{A}$ (via the constructed map); its minimality is verified by the following:

Proposition 2.2. *Let $\mathscr{M}' \to \mathscr{A}$ be a free DGA mapping to $\mathscr{A}$ inducing an isomorphism on cohomology, then $\mathscr{M}$ is isomorphic to a sub-DGA of $\mathscr{M}'$ over $\mathscr{A}$.*

Proof. First we build a DGA map from $\mathscr{M}$ to $\mathscr{M}'$ inductively; assume a map is built already on $\mathscr{M}_k$; then the next step is to map the new generators x_α, y_β into $\mathscr{M}'$. This amounts to finding x'_α, y'_β in $\mathscr{M}'$ solving the (linear differential) equations

$$dx'_\alpha = 0, \quad dy'_\beta = P_\beta(\cdots).$$

The first equation can be solved by choosing a cocycle in $\mathscr{M}'$ representing the cohomology class of x_α (here we identify the cohomologies of $\mathscr{A}, \mathscr{M}, \mathscr{M}'$). The second equation can also be solved because $P_\beta(\ldots)$ in $\mathscr{M}$ is a polynomial of lower degree generators of $\mathscr{M}$, which are mapped into $\mathscr{M}'$ already; this polynomial expression is closed in $\mathscr{M}$ and so too closed in $\mathscr{M}'$; furthermore it is exact in $\mathscr{M}$ which means it is zero in cohomology, and thus exact in $\mathscr{M}'$.

So inductively we build a DGA map $\mathscr{M} \to \mathscr{M}'$ over $\mathscr{A}$ that yields an isomorphism on cohomology by commutativity. We now inductively prove this map must be injective.

(i) The map $\mathcal{M}_2 \to \mathcal{M}'$ is injective up to degree 2; we assume the map $\mathcal{M}_k \to \mathcal{M}'$ is injective, or equivalently $\mathcal{M}_k \to \mathcal{M}'$ is injective up to degree k (since $\mathcal{M}_k$ is freely generated in degrees $\leq k$ and $\mathcal{M}'$ is free).

(ii) Up to degree k the difference between $\mathcal{M}_{k-1}$ and $\mathcal{M}_k$ is the linear span of x_α, y_β; it suffices to prove the map $\mathcal{M}_k \to \mathcal{M}$ is injective in degree k. Let us analyze modulo $\mathcal{M}_{k-1}$ by looking at the long exact sequence of the pair $(\mathcal{M}, \mathcal{M}_{k-1})$ and the corresponding pair $(\mathcal{M}', \mathcal{M}_{k-1})$ (here we identify $\mathcal{M}_k$ as a subalgebra of $\mathcal{M}$ by induction hypothesis); then it is clear that $H^k(\mathcal{M}, \mathcal{M}_{k-1})$ is isomorphic to $H^k(\mathcal{M}', \mathcal{M}_{k-1})$. The former is nothing but the span of the new generators x_α, y_β (modulo $\mathcal{M}_{k-1}$ the differential on y_β becomes zero); (exercise) this in turn implies $\mathcal{M}_k \to \mathcal{M}$ is injective up to degree k and consequently injective by freeness.

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Exercise 2.3. What should one do without the assumption $\mathscr{A}^1 = 0$?

Remark 2.4. There are choices made in building the DGA $\mathcal{M}$ such as choosing base for cohomology groups; but the above proposition in fact implies that different choices yield isomorphic results. Indeed, suppose $\mathcal{M}$ and $\mathcal{M}'$ are two such DGAs built for $\mathscr{A}$, then we have injective DGA maps

$$\mathcal{M} \to \mathcal{M}', \mathcal{M}' \to \mathcal{M}.$$

By looking at the behaviors of these maps on generators (assume both are finitely generated in each degree), we conclude that they are isomorphisms.

We shall call $\mathcal{M}(\mathscr{A})$ constructed this way <u>the</u> *minimal model* of $\mathscr{A}$. This construction of $\mathcal{M}$ can be extended to deal with non-simply-connected algebras, but further assumptions are needed so that we are not stuck at an inductive step. This will be discussed later.

Now let us analyze what can be extracted from the minimal model $\mathcal{M}$ beyond the cohomology of $\mathscr{A}$. Since $\mathcal{M}$ is a free algebra, all its information is determined by

(i) the graded vector space $V = V^2 \oplus V^3 \oplus \cdots$ of generators; and

(ii) the differential d on V; from construction d has no linear term so $d : V \to \wedge^2 V \oplus \wedge^3 V \oplus \cdots$ can be written as $d = d_2 + d_3 + \ldots$. The equation $d^2 = 0$ implies $(d_2)^2 = 0$ and further relations $\sum_{i+j=n} d_i d_j = 0$.

The quadratic part $d_2 : V \to V \wedge V$ can be dualized to give a Lie bracket on $\mathfrak{g} = V^\vee$. The cubic, quartic, quintic, ... parts of d yield a sequence maps $l_k : \wedge^k \mathfrak{g} \to \mathfrak{g}$. The equation $d^2 = 0$ implies

that $l_2 = [-,-], l_3, l_4, \ldots$ form a Lie-infinity structure on $\mathfrak{g}$ (this is the definition of a Lie-infinity structure) .

Exercise 2.5. Write down explicitly the relations among the l'_ks.

It is not hard to see that $\mathcal{M}$ is an isomorphism invariant of $\mathcal{A}$ and consequently the Lie (infinity) algebra $\mathfrak{g}$ is an isomorphism invariant of $\mathcal{A}$. These invariants cannot be derived from the cohomology of $\mathcal{A}$ as shown by the example above.

2.2 Minimal algebra

We now make an abstract definition of a minimal algebra based on the above property of minimal models. The benefit of doing so is that they become subjects of study on their own and not necessarily attached to any other DGA; moreover we shall see how to remove the technical restriction of simply-connectedness.

Definition 2.6. A connected DGA $\mathcal{A}$ is a *minimal* if it is free and its differential has no linear part, i.e. $d\mathcal{A} \subset \mathcal{A}^+ \cdot \mathcal{A}^+$.

Definition 2.7. By a minimal model of a DGA $\mathcal{A}$, we mean a minimal algebra $\mathcal{M}$ quasi-isomorphic to it.

It is natural to guess that the minimal model of a minimal DGA should be itself, because there is no redundancy in its differential. This can be proved using the proposition above (exercise), but here we pursue a more direct approach by describing it as an iterative extension. Before analyzing general minimal algebras, we briefly review the example of dual Lie algebras.

2.2.1 Dual Lie algebra

Let $\mathfrak{g}$ be a finite dimensional Lie algebra (over $\mathbb{Q}$) and

$$0 \to \mathrm{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}_{s.s.} \to 0$$

its canonical sequence associated to radical ideal (maximal solvable). Then the dual Lie algebra of $\mathfrak{g}$ is formed in two steps: first form the dual Lie algebra B of the semi-simple Lie algebra $\mathfrak{g}_{s.s.}$ whose differential is purely quadratic of the form

$$dx_k = \sum a_{ij}^k x_i \wedge x_j$$

where $\{x_k\}$ is dual to a basis of $\mathfrak{g}_{s.s.}$ and $\{a_{ij}^k\}$ the corresponding structure constants; then form an extension by adjoining new generators $\{x_\alpha\}$ dual to a basis of $\mathrm{rad}(\mathfrak{g})$ with differential of the form

$$dx_\alpha = \sum \theta_{\alpha\beta} x_\beta + a_\alpha$$

where $\Theta = (\theta_{\alpha\beta})$ and $a = (a_\alpha)$ satisfy $d\Theta - \Theta \circ \Theta = 0$ and $(d - \Theta)a = 0$. That is, Θ defines an integrable local system and a is a cocycle for these coefficients. In fact the cohomology class $[a] \in H^2(B; \Theta) \cong H^2(\mathfrak{g}_{s.s}; \Theta)$ exactly describes $\mathfrak{g}$ up to isomorphism as the extension of $\mathfrak{g}_{s.s}$ by $\mathrm{rad}(\mathfrak{g})$.

Remark 2.8. We recall the definition of twisted cohomology of a DGA $\mathscr{A}$ with twisted coefficients. The coefficient system consists of a vector space V in degree zero and a twisting matrix $V \xrightarrow{\Theta} \mathscr{A}^1 \otimes V$ satisfying the integrability condition that d_Θ given by

$$d_\Theta\{a_\alpha\} = \{da_\alpha - \sum_\beta \theta_{\alpha\beta} a_\beta\}$$

is a differential on $Hom(V, \mathscr{A})$, where $\{a_\alpha\}$ is a general element of $Hom(V, \mathscr{A})$ in terms of a basis $\{v_\alpha\}$ of V. The twisted cohomology $H^\bullet(\mathscr{A}; \Theta)$ of $\mathscr{A}$ with coefficients in V is defined to be the cohomology of the complex $(Hom(V, \mathscr{A}), d_\Theta)$. One can also define a twisted homology to be the cohomology of the complex $(\mathscr{A} \otimes V, d_\Theta)$[1].

2.2.2 General minimal algebra

Suppose now $\mathscr{A}$ is a minimal DGA and $\mathscr{A}_{k-1}$ its subalgebra generated by elements of degree $< k$. Let $\{x_\alpha\}$ be a complete set of algebraic generators in degree k. Since $d\mathscr{A} \subset \mathscr{A}^+ \cdot \mathscr{A}^+$ we can write equations:

$$dx_\alpha = \sum_\beta \theta_{\alpha\beta}^k x_\beta + a_\alpha$$

where $\Theta^k = (\theta_{\alpha\beta}^k)$ are taken from degree one and the a_α lie in $\mathscr{A}_k$. Therefore $\mathscr{A}_k$ is obtained from $\mathscr{A}_{k-1}$ by adjoining new generators subject to the above equations. We shall call such an extension a *linear extension*.

From $d^2 = 0$ one concludes (suppress upper index k for simplicity):

(i) $\Theta = (\theta_{\alpha\beta})$ is an integrable local system, i.e. $d\Theta - \Theta \circ \Theta = 0$ or $d_\Theta^2 = 0$ where $d_\Theta = d - \Theta$.

(ii) $a = \{a_\alpha\}$ is a cocycle in $\mathscr{A}_{k-1}$ for the twisting matrix Θ, i.e. $d_\Theta(a) = 0$.

(iii) a is a coboundary in $\mathscr{A}_k$ for Θ since $d_\Theta(x) = a$ where $x = \{x_\alpha\}$.

---

[1] Some prefer to call this tensor version of twisted cohomology and the previous one hom version.

Now if we change the generators $x = \{x_\alpha\}$ (in all ways) by $x' = Ax + b$ for A invertible and $b \in \mathscr{A}_{k-1}$ then we have the corresponding changes:

$$\Theta \mapsto \Theta_A = A\Theta A^{-1}, a \mapsto Aa + d_{\Theta_A} b$$

This means[2] up to isomorphism $\mathscr{A}_k$ is determined by the cohomology class of a in $H^{k+1}(\mathscr{A}_{k-1}; \Theta)$.

Theorem 2.9. *A minimal DGA $\mathscr{A} = \cup_k \mathscr{A}_k$ is determined up to isomorphism by:*

(i) *the dual Lie algebra $\mathscr{A}_1 \subset \mathscr{A}$ defined by quadratic equations:*

$$dx_k = \sum a_{ij}^k x_i \wedge x_j$$

giving d on the one-dimensional generators;

(ii) *an inductive sequence of twisted cohomology classes:*

$$a^{k+1} \in H^{k+1}(\mathscr{A}_{k-1}; \Theta^k) \quad (k = 2, 3, \dots)$$

describing $\mathscr{A}_k$ as a linear extension of $\mathscr{A}_{k-1}$.

Note that if $\mathscr{A}^1 = 0$ then $\{\Theta^k\} \subset \mathscr{A}^1$ are trivial, hence the twisted cohomology classes are ordinary cohomology classes and $\mathscr{A}$ is *structured by ordinary cohomology*. In this case, the minimal model of $\mathscr{A}$ is clearly isomorphic to $\mathscr{A}$.

In general if $\mathscr{A}$ is a free algebra filtered by a sequence of subalgebras $F_1(\mathscr{A}) \subset F_2(\mathscr{A}) \subset \dots$ so that

(i) each dimension of $\mathscr{A}$ is exhausted by some $F_k(\mathscr{A})$,

(ii) $F_k(\mathscr{A})$ is a linear extension of $F_{k-1}(\mathscr{A})$ without twisting, i.e. the structural equation $dx = \Theta x + a$ becomes $dx = a$.

Then $\mathscr{A}$ is structured by ordinary cohomology, such minimal algebras are said to be *nilpotent*.

Proposition 2.10. *A minimal DGA $\mathscr{A}$ is nilpotent if and only if its dual Lie algebra $\mathscr{A}_1$ is nilpotent and its twisting matrices $\Theta^2, \Theta^3, \dots$ are nilpotent.*

Proof. That $\mathscr{A}_1$ is nilpotent is equivalent to it can be filtered by a finite sequence of subalgebras structured by untwisted linear extensions; similarly that Θ^k is nilpotent is equivalent to that V^k can be filtered by a finite sequence of subspaces so that Θ^k acts trivially on the successive quotients; and thus one can add generators in V^k accordingly so that the extensions are untwisted. ∎

---

[2] Θ and Θ_A correspond to isomorphic linear maps.

2.2.3 Factorization of free algebra

So far we have a rather complete understanding of minimal algebras, i.e. free algebras equipped with higher order (quadratic and more) differentials; it splits into a purely quadratic part followed by linear extensions structured by certain twisted cohomology classes.

Now we consider the complementary situation where the differential only has linear part. In this case $d : V \to V$ decomposes V into subspaces H and C on which d acts by zero and isomorphism respectively. Consequently the free algebra in consideration factors into a free algebra with trivial differential (which is in particular minimal) and a free algebra with trivial cohomology, i.e. *contractible*. The same argument proves that every free algebra can be factored into a contractible one and a minimal one, respectively corresponding to the linear and higher order parts of its differential; this is clear considering multiplication only, the subtlety lies in dealing with differential.

Proposition 2.11. *Every connected free DGA factors into a contractible one and a minimal one in a unique way.*

Proof. The differential d in $\mathscr{A}$ induces a differential $d' : V \to V$ on the graded vector space of generators or indecomposables; it decomposes V into $(C, d' = iso.) \oplus (H, d' \equiv 0)$ as before. We lift C to generate a contractible subalgebra $\mathscr{C} \subset \mathscr{A}$ and let $\mathscr{M}$ be the quotient $\mathscr{A}/\text{ideal}\mathscr{C}^+$ which is our desired minimal algebra. Since $\mathscr{A}$ is free, $\mathscr{A} \simeq \mathscr{C} \otimes \mathscr{M}$ as algebras, but $\mathscr{M}$ need not be closed under d; in order to improve this into a DGA isomorphism, we need to produce a DGA splitting $\mathscr{M} \to \mathscr{A}$.

Now we apply the argument of Proposition 2.2. Since $\mathscr{M}$ is minimal, getting such a splitting amounts to inductively solving a sequence of linear differential equations according to the structure of $\mathscr{M}$; it suffices to prove:

(i) (the image of) $\mathscr{M}_1$ is closed under d; and

(ii) the structural cocycles $a^3, a^4, \ldots$ of $\mathscr{M}$ are exact in $\mathscr{A}$.

The former can be achieved by choosing (the image of) $\mathscr{M}_1$ to be the subalgebra of $\mathscr{A}$ generated by $d^{-1}(\wedge^2 \mathscr{A}_1)$; the latter follows from that $\mathscr{A} \to \mathscr{M}$ induces an isomorphism on cohomology for all twisted coefficients whose proof is left to the reader, the following lemma should come handy. See also Lemma 2.17. $\blacksquare$

Lemma 2.12. *Let $\mathscr{A} \xrightarrow{s} \mathscr{A}$ be a contractible homotopy, i.e. $ds + sd \equiv id$. Let $\mathscr{O}_s$ denote the subspace of $\mathscr{A}_1$ whose left multiplication operations anti-commutes with s. Then the twisted cohomology of $\mathscr{A}$ vanishes if the twisting matrix has entries in $\mathscr{O}_s$.*

Proof. Abusing the notation, we still denote by s the induced map of degree -1 on $Hom(V, \mathscr{A})$. Then we have $\Theta s + s\Theta = 0$ because the entries of Θ lie in $\mathscr{O}_s$. Then $d_\Theta s + s d_\Theta = ds + sd = id$. $\blacksquare$

2.3 Homotopy between DGAs

At this point we see striking analogies between minimal DGAs and topological spaces.

Space	DGA
fundamental group	dual Lie algebra
k-th homotopy group	space of k-dimensional generators
action of fundamental group on higher homotopy	twisting matrices
cohomological k-invariants for Postnikov system	structural cohomology classes
Whitehead brackets	Lie brackets on dual space to generators
. . .	?

From this comparison it is reasonable to conjecture that

(i) there is an appropriate notion of homotopy for DGAs, and minimal models should play the role of Postnikov systems for spaces;

(ii) there is an appropriate notion of rational homotopy equivalence for spaces so that spaces up to rational homotopy equivalence are equivalent to rational DGAs up to (DGA) homotopy equivalence.

In this section we treat (i), while we postpone the treatment of (ii) to the next chapter after developing enough algebraic tools for DGAs.

2.3.1 First definition of homotopy

We give a definition of homotopy between DGA maps mimicking the topological one that a homotopy between $f, g : X \to Y$ is a map $H : X \to Y^I$ whose evaluations at $0, 1$ recover f, g.

Definition 2.13. Let f, g be DGA maps from $\mathscr{A}$ to $\mathscr{C}$. A homotopy from f to g is a DGA map:

$$H : \mathscr{A} \to \mathscr{C} \otimes \wedge(t, dt)$$

satisfying $H|_{t=0} = f$ and $H|_{t=1} = g$. If such H exists, we say f is homotopic to g. The homotopy classes of maps will be denoted by $[\mathscr{A}, \mathscr{C}]$.

This definition clearly mirrors the usual topological definition of homotopy $X \times I \to Y$; the algebra $\wedge(t, dt)$ (in which $|t| = 0$) plays the role of the interval. The theorem we aim to prove is:

Theorem 2.14. *Let $\mathcal{M}(\mathscr{A}) \to \mathscr{A}$, $\mathcal{M}(\mathscr{B}) \to \mathscr{B}$ be <u>nilpotent</u> minimal models of $\mathscr{A}$ and $\mathscr{B}$ respectively. Then every DGA map $\mathscr{A} \to \mathscr{B}$ admits a <u>homotopy lifting</u> $\mathcal{M}(\mathscr{A}) \to \mathcal{M}(\mathscr{B})$, i.e. the diagram*

$$
\begin{array}{ccc}
\mathcal{M}(\mathscr{A}) & \longrightarrow & \mathcal{M}(\mathscr{B}) \\
\downarrow & & \downarrow \\
\mathscr{A} & \longrightarrow & \mathscr{B}
\end{array}
$$

commutes up to homotopy; such a homotopy lifting is unique up to homotopy. In particular $[\mathscr{A}, \mathscr{B}]$ is in bijection with $[\mathcal{M}(\mathscr{A}), \mathcal{M}(\mathscr{B})]$.

Proof. This follows quite formally from the proposition below. ∎

Proposition 2.15 (homotopy lifting). *Given the diagram of DGA maps*

$$
\begin{array}{ccc}
 & & \mathscr{A}' \\
 & & \downarrow{\scriptstyle \pi} \\
\mathcal{M} & \xrightarrow{\;f\;} & \mathscr{A}
\end{array}
$$

with $\mathcal{M}$ <u>nilpotent</u> minimal and π inducing an isomorphism on cohomology.

(i) *There exists a homotopy lifting $f' : \mathcal{M} \to \mathscr{A}'$, i.e. $\pi f'$ is homotopic to f. If further π is surjective, then we can choose f' so that $\pi f' = f$.*

(ii) *Such a homotopy lifting f' is unique up to homotopy. In particular π induces a bijection $[\mathcal{M}, \mathscr{A}'] \to [\mathcal{M}, \mathscr{A}]$.*

Proof. The nilpotency condition allows one to reduce to the inductive step of (untwisted) linear extension; this proposition thus follows from the theorem below which shows that the obstructions for finding a lifting and a homotopy are both controlled by cohomology. ∎

Theorem 2.16 (obstruction theory). *Let $\mathscr{A}$ be a (possibly twisted) linear extension of $\mathscr{B}$, the linear space of new generators will be called the <u>dual homotopy</u> of $(\mathscr{A}, \mathscr{B})$. Then*

(i) *(Existence) the obstruction for extending a map $f : \mathscr{B} \to \mathscr{C}$ to $\mathscr{A}$ lies in the cohomology of $\mathscr{C}$ with twisted coefficients in the dual homotopy of $(\mathscr{A}, \mathscr{B})$.*

(ii) *(Uniqueness) the obstruction for extending a homotopy $H : \mathscr{B} \to \mathscr{C} \otimes \wedge(t, dt)$ between $f|_{\mathscr{B}}$ and $g|_{\mathscr{B}}$, where f and g are maps $\mathscr{A} \to \mathscr{C}$, lies in the cohomology of $\mathscr{C}$ with twisted coefficients in the dual homotopy of $(\mathscr{A}, \mathscr{B})$.*

Proof. The proof of (i) is the same as that of Proposition 2.2 and Theorem 2.9; we point out that if the dual homotopy V of $(\mathscr{A}, \mathscr{B})$ concentrates in dimension k, then the obstruction lies in (twisted cohomology) $H^{k+1}(\mathscr{C}; V)$.

Now to prove (ii), we write the structure equation $dx = \Theta x + a$ as before and solve for $H(x)$ satisfying $dH(x) = H(\Theta)H(x) + H(a)$ and boundary conditions $H(x)|_{t=0,1} = f(x), g(x)$. For simplicity we assume $\Theta = 0$, which is already sufficient for our applications. Consider the integration operator $\int_0^t$ which has the property that

$$d \int_0^t + \int_0^t d = \mathrm{id} - \mathrm{ev}_0,$$

and in particular

$$d \int_0^1 + \int_0^1 d = \mathrm{ev}_1 - \mathrm{ev}_0.$$

Now suppose $H(x)$ is the desired solution, then one has

$$d \int_0^1 H(x) = g(x) - f(x) - \int_0^1 H(dx). \quad \text{(using } Hd = dH\text{)}$$

Notice that since dx lies in $\mathscr{B}$, the right hand side is fixed by the data given; moreover it is d-closed since

$$d \int_0^1 H(dx) + \int_0^1 dH(dx) = g(dx) - f(dx).$$

One sees that the cohomology class of the right hand side is the obstruction to finding $\int_0^1 H(x)$ and consequently is an obstruction for finding $H(x)$. It turns out that this is the only obstruction. Indeed, once we have $\int_0^1 H(x)$, it is easy to verify that

$$H(x) := f(x) + \int_0^t H(dx) + d\left(t \cdot \int_0^1 H(x)\right).$$

gives a desired solution.

The general case $\Theta \neq 0$ can be proved by producing an operator $\tilde{\int}$ satisfying

$$d_\Theta \tilde{\int}_0^t + \tilde{\int}_0^t d_\Theta = \mathrm{id} - \mathrm{ev}_0.$$

Such an operator exists because $\mathrm{ev}_0 : \mathscr{C} \otimes \wedge(t, dt) \to \mathscr{C}$ induces an isomorphism on cohomology for all twisted coefficients. A similar fact has been used in Proposition 2.11, but the full proof was not given; below we detail the proof. ∎

Lemma 2.17. $\mathscr{C} \otimes \wedge(t, dt) \xrightarrow{t=0} \mathscr{C}$ *induces an isomorphism on cohomology for all twisted coefficients.*

Proof. Treat $\mathscr{C} \otimes \wedge(t, dt)$ as a module over $\wedge(t, dt)$ and filter $\mathscr{C} \otimes \wedge(t, dt) \supset \mathscr{I} \supset \mathscr{I}^2 \supset \ldots$ by a sequence of submodules corresponding to powers of t; the successive quotients $\mathscr{I}^k/\mathscr{I}^{k+1}$ are isomorphic to $\mathscr{C} \otimes \text{span}\{t^k, t^{k-1}dt\}$ with product differential. It is easy to define a contracting homotopy for $\mathscr{I}^k/\mathscr{I}^{k+1}$ commuting with the operation of left multiplication by $\mathscr{C}$. By Lemma 2.12, $\mathscr{I}^k/\mathscr{I}^{k+1}$ is acyclic for all twisted coefficients. Note that passing to successive quotients, the action of the twisting matrix $\Theta(t)$ becomes a constant one by $\Theta(0)$. Now any twisted cohomology of $\mathscr{C} \otimes \wedge(t, dt)$ comes with an induced filtration whose successive quotients are zero; except for the very first layer $\mathscr{C} \otimes \wedge(t, dt)/\mathscr{I} \cong \mathscr{C}$. This proves the desired isomorphism provided the induced filtration on cohomology is exhaustive, e.g. cohomology is finite dimensional in each degree, or $|t| > 0$. $\blacksquare$

We see that, the technical difficulty in proving obstruction theory for twisted coefficients comes from that the definition of homotopy does not cooperate with local coefficient systems in $\mathscr{A}$; the differential d_Θ instead of d is the more natural one in homotopy extension problem. This consideration leads us to the second definition of homotopy which appears to be less natural at first glance but turns out to be much easier to work with.

2.3.2 Second definition of homotopy

The second definition of homotopy mimics the topological one expressed by $X \to Y^I$. Let us write $\mathscr{A} = \mathscr{B}(x_\alpha)$ be a (successive) linear extension of $\mathscr{B}$, meaning that

$$dx_\alpha = \sum_\beta \theta_{\alpha\beta} x_\beta + a_\alpha, \quad |x_\alpha| > 0$$

where $\{\theta_{\alpha\beta}\}$ and $\{a_\alpha\}$ belong to earlier stages. Here $a_\alpha = P_\alpha(x)$ is a polynomial of earlier generators x with coefficients in $\mathscr{B}$.

Write $\mathscr{A} \otimes_\mathscr{B} \mathscr{A} = \mathscr{B}(x_\alpha, y_\alpha)$, and $\mathscr{A}^I = \mathscr{B}(x_\alpha, y_\alpha, \overline{\delta}_\alpha)$ where $\overline{\delta}_\alpha$ is a new generator of degree one less than

$$\delta_\alpha = \delta(x_\alpha, y_\alpha) = x_\alpha - y_\alpha.$$

To make $\mathscr{A}^I$ into a DGA, we need to specify differential on $\overline{\delta}_\alpha$, which naively should interpolate x_α and y_α, i.e.

$$d\overline{\delta}_\alpha = \delta_\alpha.$$

However this is problematic in two ways: first from the experience we had so far, it seems more natural to replace d by d_Θ as it is the cohomology of the latter that controls the structure and inductive obstructions of homotopy; second

$$d_\Theta \delta_\alpha = P_\alpha(x) - P_\alpha(y)$$

is zero modulo earlier terms but not zero on the nose. So the best we can aim for is to find a differential so that

$$d_\Theta \overline{\delta}_\alpha = \delta_\alpha + \text{earlier terms.}$$

Suppose for now such a differential can be found, then we define

Definition 2.18. Two maps of $\mathscr{A}$ into $\mathscr{C}$ extending $\mathscr{B} \xrightarrow{\nu} \mathscr{C}$ are homotopic (rel ν) if the combined map of $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}$ into $\mathscr{C}$ extends over $\mathscr{A}^I$. The equivalence classes of maps (rel ν) up to homotopy will be denoted by $[\mathscr{A}, \mathscr{C}; \nu]$.

This definition makes it trivial that the obstructions to finding a homotopy is exactly the obstructions to solving the equation $d_\Theta \overline{\delta}_\alpha = \delta_\alpha + \text{earlier terms}$ in the target. Now to complete this definition we show

Lemma 2.19. *We can define a differential on $\mathscr{A}^I = \mathscr{B}(x_\alpha, y_\alpha, \overline{\delta}_\alpha)$ extending that on $\mathscr{B}(x_\alpha, y_\alpha)$ of the form*

$$d_\Theta \overline{\delta}_\alpha = \delta_\alpha + \text{earlier terms}$$

such that the natural map $\mathscr{A}^I \to \mathscr{A}$ sending δ_α and $\overline{\delta}_\alpha$ to zero induces an isomorphism on cohomology for all coefficients in $\mathscr{B}$.

Proof. The proof goes by induction on the new generators using the vanishing of the cohomology for the ideal $(\overline{\delta}_\alpha, \delta_\alpha)$. By induction we can solve for η_α in earlier terms the equation

$$d_\Theta \eta_\alpha = P_\alpha(x) - P_\alpha(y)$$

because the right hand side belongs to the ideal of earlier generators. Then we can define

$$d_\Theta \overline{\delta}_\alpha = \delta_\alpha + \eta_\alpha.$$

Finally the DGA isomorphism (with respect to $d_{\Theta'}$)

$$\mathscr{A}^I \cong \mathscr{B}(x_\alpha) \otimes \text{algebra}(\overline{\delta_\alpha}, d_{\Theta'} \overline{\delta}_\alpha)$$

implies the new ideal $(\overline{\delta}_\alpha, \delta_\alpha) = (\overline{\delta}_\alpha, d_{\Theta'} \overline{\delta}_\alpha)$ has vanishing cohomology for an arbitrary coefficient system Θ'. ∎

Remark 2.20. Note that different choices of η yield changes $\overline{\delta} \to \overline{\delta} + \text{decomposable}$ from the ideal. Therefore the homotopy relation is independent of the choice of d in $\mathscr{A}^I$ (mod decomposables).

As applications of this new definition, we prove

Corollary 2.21. *Homotopy is an equivalence relation, and we can add homotopies.*

Proof. With obvious notations, we use obstruction theory to extend the inclusion:

$$\mathscr{B}(x_\alpha, z_\alpha) \to \mathscr{B}(x_\alpha, y_\alpha, z_\alpha, \overline{\delta}(x_\alpha, y_\alpha), \overline{\delta}(y_\alpha, z_\alpha))$$

over $\mathscr{B}(x_\alpha, z_\alpha, \overline{\delta}(x_\alpha, z_\alpha))$ using the vanishing of the cohomology of:

$$\mathrm{ideal}(\overline{\delta}(x_\alpha, y_\alpha), \overline{\delta}(y_\alpha, z_\alpha), \delta(x_\alpha, y_\alpha), \delta(x_\alpha, y_\alpha)).$$

This proves the transitivity by adding homotopies. Symmetry follows by transposing d by the involution:

$$x_\alpha \mapsto y_\alpha, \ y_\alpha \mapsto x_\alpha, \ \overline{\delta}_\alpha \mapsto -\overline{\delta}_\alpha.$$

∎

Corollary 2.22. *Homotopic maps induce the same map on cohomology.*

Proof. It suffices to prove the natural inclusions $\mathscr{A} \hookrightarrow \mathscr{A}^I$ through x_α or y_α induce the same map oh cohomology. Indeed either map, when post-composed with the canonical projection $\mathscr{A}^I \to \mathscr{A}$, becomes the identity map. ∎

Exercise 2.23. Prove the same under the first definition.

Proposition 2.24. *Assume that the dual homotopy of $(\mathscr{A}, \mathscr{B})$ is finite dimensional and that $\mathscr{C}$ is finite dimensional in each degree of an generator in dual homotopy and one less. Then the set of homotopy class of maps $[\mathscr{A}, \mathscr{C}; \nu]$ has the structure of of an affine algebraic variety modulo an equivalence relation which is the image of an affine algebraic variety.*

Proof. Consider the array:

$$Hom(\mathscr{B}(x_\alpha, y_\alpha, \overline{\delta}_\alpha), \mathscr{C}) \to Hom(\mathscr{B}(x_\alpha), \mathscr{C}) \times Hom(\mathscr{B}(y_\alpha), \mathscr{C}) \to [\mathscr{A}, \mathscr{C}; \nu].$$

Each of the hom-sets are affine algebraic since the conditions imposed by commuting with d are algebraic. ∎

Example 2.25 (cohomology and homotopy are representable). (i) Let $\mathscr{K}_n$ be the free algebra on one generator in dimension n with $d \equiv 0$. Then $Hom(\mathscr{K}_n, \mathscr{A}) = \ker d$, and

$$[\mathscr{K}_n, \mathscr{A}] = H^n(\mathscr{A}).$$

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(ii) Let $\mathscr{A}_{S^n}$ be any DGA with cohomology isomorphic to that of the n-sphere, and $\mathscr{M}$ a minimal algebra. Then by obstruction theory $[\mathscr{M},\mathscr{A}_{S^n}] = [\mathscr{M}_n,\mathscr{A}_{S^n}]$; indeed the obstructions for extending a map and a homotopy from $\mathscr{M}_n \to \mathscr{A}_{S^n}$ to $\mathscr{M} \to \mathscr{A}_{S^n}$ vanish. Similarly we can apply homotopy to assume the map on $\mathscr{M}_{n-1}$ is trivial and thus $[\mathscr{M}_n,\mathscr{A}_{S^n}] = [\mathscr{M}_n,\mathscr{A}_{S^n};\mathrm{rel}\,\mathscr{M}_{n-1}]$. That said, we can assume $\mathscr{M} = \wedge(V_n)$ on which $d \equiv 0$. Then from the previous example we see that

$$[\mathscr{M},\mathscr{A}_{S^n}] = \text{dual of } V_n = n\text{-th homotopy of } \mathscr{M}.$$

2.4   Minimal model, revisit

Let $\mathscr{B} \to \mathscr{A}$ be a DGA map, and Σ a collection of coefficients in $\mathscr{B}$ for which $H^0\mathscr{B} = H^0\mathscr{A}$ and $H^1\mathscr{B} \to H^1\mathscr{A}$ is injective.

Theorem 2.26 (Relative minimal model). *We can minimally extend $\mathscr{B} \to \mathscr{A}$ to $\mathscr{B}(x_\alpha) \xrightarrow{f} \mathscr{A}$ so that f induces an isomorphism of Σ-cohomology. Any two such extensions are isomorphic; the isomorphisms are well-defined up to homotopy over $\mathscr{A}$.*

Proof. The construction of $\mathscr{B}(x_\alpha)$ is the same as before, we add generators to generate (twisted) cohomology and add more generators to kill relations. Its uniqueness follows from obstruction theory with twisted coefficients. ∎

Definition 2.27. We call $\mathscr{B}(x_\alpha)$ the minimal model of $\mathscr{A}$ over $\mathscr{B}$ relative to the coefficients Σ.

Example 2.28 (de Rham algebra).   (i) (odd sphere) $\wedge(x)$, $|x|$ odd, $dx = 0$.

  (ii) (even sphere) $\wedge(x,y)$, $|x|$ even, $dy = x^2$.

  (iii) (complex projective space) $\wedge(x,y)$, $|x| = 2$, $dy = x^{n+1}$.

  (iv) (Lie group) $\wedge(x_1,x_2,\ldots,x_n)$, $|x_i|$ odd, $dx_i = 0$.

  (v) (Riemann surface of genus $g > 1$) $\wedge(x_1,y_1,\ldots,x_g,y_g;\ldots;\ldots)$, the number of generators needed in the n-th chunk is about size g^n.

  (vi) (Hirzebruch surface) $\wedge(x,y;z,w)$, $|x| = |y| = 2$, $dz = x^2$, $dw = y^2 + nxy$.

Example 2.29 (twisted coefficients). Let $\mathscr{B}$ be the dual Lie algebra of a semi-simple Lie algebra. Then $H^1\mathscr{B}$ and $H^2\mathscr{B} = 0$ for all finite-dimensional coefficients Σ (see Whitehead). So any $\mathscr{B} \to \mathscr{A}$ can be extended to a model for any collection of finite coefficients Σ in $\mathscr{B}$.

Example 2.30. (formal computation)

(i) Suppose $\mathscr{B} \to \mathscr{A}$ is given so that $H\mathscr{A}$ is concentrated in degrees $\leq n$ and the map induces an isomorphism on cohomology up to degree n, then the computation of minimal model becomes completely formal in that the target $\mathscr{A}$ is irrelevant.

(ii) (4-manifold) Let M^4 be a compact simply-connected 4-manifold. Then its minimal model is formally determined by its intersection form on H^2. Compare with Freedman's theorem that the intersection form determines the topological type of a smooth simply connected 4-fold.

(iii) (formal DGA) If $\mathscr{A}$ is quasi-isomorphic (as DGA) to its cohomology $H\mathscr{A}$, then building a minimal model for $\mathscr{A}$ is the same as building one for its cohomology; the computation is formal in that the actual structure of $\mathscr{A}$ and its differential but its cohomology are irrelevant.

(iv) (geometrically formal manifold) If M is a Riemannian manifold on which products of harmonic forms remain harmonic, then its de Rham algebra of differential forms is formal. Examples include all compact symmetric spaces.

2.5 Automorphism

The automorphism group of a nilpotent minimal algebra $\mathscr{A}$ is conceptually easy to understand, it maps generators to generators to give an algebra isomorphism; and it further commutes with d. If $\mathscr{A}$ is finitely generated, then its automorphism group is clearly an *algebraic matrix group* as the conditions posed by commuting with d are algebraic.

Example 2.31. Consider $\mathscr{A} = \wedge(x_1, \ldots, x_n; dx_i = 0, dy = x_1^2 + \cdots + x_n^2)$ where $|x_i| = 2$. An automorphism σ of $\mathscr{A}$ should

(i) be an algebra isomorphism; that means σx_i is a linear combination of $x_i's$ and σy is Const.$y +$ decomposables, but for obvious degree reason $\sigma y = $ Const.y;

(ii) commute with d; that means $d(\sigma x) = \sigma(dx) = 0$ (which is vacuous in our example) and $d(\sigma y) = \sigma(dy)$ i.e.

$$p(\sigma x) = \text{Const.} p(x) \quad \text{where } p(x) = x_1^2 + \cdots + x_n^2.$$

This automorphism group is the so-called orthogonal similitude group, which is a reductive group.

Example 2.32. $\mathscr{A} = \wedge(x_1, \ldots, x_n; y_1, \ldots, y_r)$ where $|x_i| = 2$, $dx_i = 0$ and $dy_j = p_j(x_1, \ldots, x_n)$ in which $p_1, p_2, \ldots, p_r$ form a regular sequence in $\wedge(x_1, \ldots, x_n)$. Ignoring products with multiplicative group, all connected algebraic groups occur as the automorphism group of such an $\mathscr{A}$.

We are more interested in the automorphism group modulo homotopy equivalence, i.e. homotopy classes of automorphisms; we shall call these *outer* automorphisms and call those automorphisms homotopic to identity *inner*. From the previous section, the automorphism group of an arbitrary DGA modulo homotopy equivalence is identified with outer automorphism group of its minimal model.

We shall also look at and compare the induced automorphisms of cohomology, the dual homotopy (space of indecomposables) and the spherical cohomology (cocycles modulo indecomposables, analogous to image of Hurewicz homomorphism) and call these homology automorphisms, homotopy automorphisms and spherical homology automorphisms respectively.

The (rational) algebraic nature of these automorphisms allows us to study them through their "infinitesimal behavior" following the general scheme:

$$
\begin{array}{lcccccc}
\text{Lie algebra:} & & \text{alg. Der.} & \supset & \text{DGA Der.} & \supset & \text{inn. Der.} \\
& \log \Big\uparrow \Big\downarrow \exp & & & & & \\
\text{Lie/Algebraic group:} & & \text{alg. Aut.} & \supset & \text{DGA Aut.} & \supset & \text{inn. Aut.}
\end{array}
$$

Since there is no analytic structure involved, in order to avoid infinite sum in exponential and logarithm, we need to impose nilpotency condition on derivation and unipotency condition on automorphism. The following characterizes when an automorphism is unipotent.

Proposition 2.33. *An automorphism of a nilpotent minimal algebra is unipotent if and only if it acts unipotently on either one of the following: dual homotopy $\pi = \mathscr{A}/(\mathscr{A}^+ \cdot \mathscr{A}^+)$, spherical cohomology $\Sigma = \ker d/(\ker d \cap \mathscr{A}^+ \cdot \mathscr{A}^+)$ and cohomology $H = \ker d/\operatorname{im} d$.*

Proof. This follows by induction from the short exact sequences

(i) $0 \to \mathscr{A}^+ \cdot \mathscr{A}^+ \to \mathscr{A} \to \pi \to 0$,

(ii) $0 \to \Sigma \to \pi \to \mathscr{A}/(\ker d + \mathscr{A}^+ \cdot \mathscr{A}^+) \to 0$, and

(iii) $0 \to \mathscr{A}^+ \cdot \mathscr{A}^+/\operatorname{im} d \to H \to \Sigma \to 0$.

The action of an automorphism on $\mathscr{A}^+ \cdot \mathscr{A}^+$, $\mathscr{A}/(\ker d + \mathscr{A}^+ \cdot \mathscr{A}^+)$ and $\mathscr{A}^+ \cdot \mathscr{A}^+/\operatorname{im} d$ are determined by its action on previous generators. $\blacksquare$

Corollary 2.34. *Inner automorphisms are unipotent.*

Proof. They act trivially on cohomology. ∎

Theorem 2.35. *The symmetry groups associated to a nilpotent minimal algebra $\mathscr{A}$:*

  (i) all automorphisms,

 (ii) outer automorphisms (homotopy classes),

(iii) homology automorphisms,

 (iv) homotopy automorphisms,

  (v) spherical automorphisms,

differ from one another only by normal unipotent subgroups. If furthermore $\mathscr{A}$ is finitely generated (i.e. π is finite dimensional) then each group above is naturally an algebraic matrix group and each has the same reductive part.

Proof. The first assertion directly follows from the proposition above, and the second follows from the short exact sequences used in its proof. ∎

Remark 2.36. An algebraic matrix group G has a Levi decomposition (over $\mathbb{Q}$) $G = G_{red} \ltimes U$ where U is the unipotent radical (maximal connected unipotent normal subgroup) and $G_{red} = G/U$ the corresponding reductive quotient. Two Levi decompositions differ by an inner automorphism from U. Examples of reductive groups include general linear group, special linear group, orthogonal, unitary, symplectic and all semi-simple matrix groups. Examples of unipotent groups include additive group, Heisenberg group and the group of all upper triangular unipotent matrices.

Remark 2.37. The Lie algebras of the inner and outer automorphism groups can be described.

  (i) the Lie algebra of inner automorphisms constitute inner derivations, i.e. those of the form $[d,i] = di + id$ for some degree -1 derivation i; see Sullivan [**infinitesimal**].

 (ii) the Lie algebra of outer automorphisms can be identified with André-Quillen cohomology $H^0_{AQ}(\mathscr{A}, \mathscr{A})$ (with Gerstenhaber bracket); see Block-Lazarev [**block-lazarev**].

Proposition 2.38. *If $\mathscr{A}$ be a nilpotent minimal algebra of finite type (i.e. finitely generated in each dimension) and that its cohomology vanishes above certain dimension. Then its <u>outer</u> automorphism group is an algebraic matrix group.*

Proof. Let $\mathscr{A}_k$ be the sub-dga of $\mathscr{A}$ generated by generators in dimensions $\leq k$. Then the outer automorpshim of $\mathscr{A}_k$ is an algebraic matrix group. Now we claim that the outer automorphism group of $\mathscr{A}$ consides with that of $\mathscr{A}_k$ for k sufficiently large. Indeed, the canonical restriction homomorrphism

$$\text{outer aut. } \mathscr{A} \to \text{outer aut. } \mathscr{A}_k$$

is both surjective and injective since the obstructions for lifting a map and a homotopy, which lie in cohomology of $\mathscr{A}$ with coefficients in relative homotopy of $(\mathscr{A}, \mathscr{A}_k)$, vanish for obvious degree reasons. ∎

Chapter 3

Rational homotopy theory

In this chapter, we discuss the relation between the homotopy theory of topological spaces and that of DGAs. We shall see that the rational part of the homotopy information of a space is completely captured by its DGA of differential forms; the Postnikov system of a space, tensored with $\mathbb{Q}$, is given by its minimal model of rational polynomial forms. Moreover we desire to build an equivalence

topological spaces/rational homotopy equivalence $\sim$ rational DGAs/homotopy equivalence

For technical reasons, we will restrict ourselves to nilpotent DGAs and nilpotent spaces. By a nilpotent DGA we mean one which has a nilpotent minimal model.

3.1 Nilpotent space

Definition 3.1. A (connected pointed) space X is *nilpotent* if $\pi_1 X$ is a nilpotent group and $\pi_n X$ is a nilpotent $\pi_1 X$-module for all n.

Remark 3.2. Let G be a π-group, consider the following *lower central series* of normal π-subgroups

$$G = \Gamma_1 G \supset \Gamma_2 G \supset \Gamma_3 G \supset \ldots$$

in which $\Gamma_2 G$ is the normal subgroup generated by $(x.g)g^{-1}$ (or $x.g - g$ if G is abelian), $x \in \pi, g \in G$; and $\Gamma_k = \Gamma_2 \Gamma_{k-1}$. Then G is π-nilpotent if $\Gamma_k M = 0$ for some k; π is a nilpotent group if π is nilpotent over itself under conjugation.

  Note that π acts trivially on the successive quotients Γ_{k-1}/Γ_k and thus π is an inductive extension by trivial π-modules; in particular π itself is an inductive *central* extension by abelian groups.

The groups $N_k := \Gamma_1/\Gamma_k$ fit in a tower

$$\cdots \to N_k \to N_{k-1} \to \cdots \to N_3 \to N_2 \to N_1 = \mathsf{e}$$

with inductive short exact sequences

$$1 \to \Gamma_{k-1}/\Gamma_k \to N_k \to N_{k-1} \to 1.$$

Theorem 3.3 (Nilpotent Whitehead theorem)**.** *Let $X \to Y$ be a continuous map between nilpotent spaces that induces an isomorphism on homology $H_*(X;\mathbb{Z}) \xrightarrow{\cong} H_*(Y;\mathbb{Z})$. Then it also induces an isomorphism on homotopy $\pi_* X \xrightarrow{\cong} \pi_* Y$.*

3.1.1 Stallings' theorem

Let us first prove a special case of the theorem.

Theorem 3.4 (Stallings)**.** *Let $\pi \to \pi'$ be a homomorphism of nilpotent groups inducing an isomorphism on group homology $H_*(\pi;\mathbb{Z}) \cong H_*(\pi';\mathbb{Z})$. Then the map is a group isomorphism.*

It will be clear from our proof that it suffices to require the induced map on homology is an isomorphism on H_1 and an epimorphism on H_2.

Proof. One inductively proves that the induced map $\pi/\Gamma_k\pi \to \pi'/\Gamma_k\pi'$ is an isomorphism for all k. The inductive step follows from the (functorial) exact sequence

$$H_2(\pi) \to H_2(\pi/\Gamma_{k-1}\pi) \to \Gamma_{k-1}\pi/\Gamma_k\pi \to H_1(\pi) \to H_1(\pi/\Gamma_{k-1}) \to 0$$

by five lemma. This exact sequence is the one from the lemma below for the pair $(\pi, \Gamma_{k-1}\pi)$; note that $\Gamma_k\pi = [\pi, \Gamma_{k-1}\pi]$. ∎

Lemma 3.5 (inflation-restriction exact sequence)**.** *Let N be a normal subgroup of G. Then there is an exact sequence*

$$H_2(G) \to H_2(G/N) \to H_0(G/N;H_1(N)) \to H_1(G) \to H_1(G/N) \to 0.$$

Proof. This is the five-term exact sequence in low dimensions associated to the Lyndon-Hochschild-Serre spectral sequence

$$H_*(G/N;H_*(N)) \Rightarrow H_*(G).$$

We note that $H_0(G/N;H_1(N)) = N/[G,N]$; indeed $H_1(N) = N/[N,N]$ on which G/N acts by conjugation hence the coinvariants are exactly $N/[G,N]$. ∎

Remark 3.6. The five term exact sequence takes the form

$$H_2 \to E^2_{2,0} \xrightarrow{d_2} E^2_{0,1} \to H_1 \to E^2_{1,0} \to 0.$$

Remark 3.7. Stallings' theorem in fact proves that any group homomorphism inducing an homology isomorphism must yield an isomorphism on $N_\infty = \varprojlim N_k$. This means $\Gamma_\infty = \varinjlim \Gamma_k$ is "phantom" to group homology and thus provides an extra invariant.

3.1.2 Proof of Theorem 3.3

We now proceed to prove the nilpotent Whitehead theorem. Let $f : X \to Y$ be a continuous map between connected nilpotent spaces inducing an isomorphism on homology. Let us first show that f induces an isomorphism on π_1. Indeed the five-term exact sequence

$$H_2(X) \to H_2(\pi_1 X) \to 0 \to H_1(X) \to H_1(\pi_1 X) \to 0$$

associated to the Serre spectral sequence for the fiberation

$$\widetilde{X} \to X \to K(\pi_1 X, 1)$$

implies that the map $\pi_1 X \xrightarrow{f_*} \pi_1 Y$ yields an isomorphism on H_1 and an epimorphism on H_2 so that Stallings' theorem applies.

Assume now that $f : X \hookrightarrow Y$ is a cofiberation, so $H_*(Y, X) = 0$. We prove inductively $\pi_*(Y, X) = 0$. Let $\pi = \pi_1 X = \pi_1 Y$ the latter identified via f; assume $\pi_i(Y, X) = 0$ for $i < n$. One sees from the exact sequence

$$\pi_n Y \to \pi_n(Y, X) \to \pi_{n-1}(X)$$

that $\pi_n(Y, X)$ is an nilpotent π-module, i.e. $I^r \pi_n(Y, X) = 0$ for some r where I is the augmentation ideal of $\mathbb{Z}[\pi]$. On the other hand, from Hurewize theorem applied to universal covering spaces we see

$$\pi_n(Y, X)/\pi\text{-action} = H_n(Y, X) = 0,$$

and thus $\pi_n(Y, X) = I\pi_n(Y, X) = \cdots = I^r \pi_n(Y, X) = 0$.

3.2 Rational homotopy equivalence

3.2.1 Rationalization of nilpotent groups

Definition 3.8. A group homomorphism between nilpotent groups is a rational isomorphism if it induces an isomorphism on rational homology $H_*(-; \mathbb{Q})$.

From the proof of Stallings' theorem, rational isomorphism implies an isomorphism on $\Gamma_*/\Gamma_{*-1} \otimes \mathbb{Q}$. This suggests the following construction.

Lemma 3.9. *Nilpotent groups can be functorially "tensored with $\mathbb{Q}$", $\pi \to \pi_{\mathbb{Q}}$, such that*

(i) $H_*(\pi; \mathbb{Q}) \xrightarrow{\cong} H_*(\pi_{\mathbb{Q}}; \mathbb{Q})$,

(ii) $H_*(\pi_{\mathbb{Q}}; \mathbb{Z})$ *is rational and that $H_*(\pi; \mathbb{Q}) \to H_*(\pi_{\mathbb{Q}}; \mathbb{Z})$ is tensoring with $\mathbb{Q}$.*

Proof. One inductively tensor the central extension

$$1 \to \Gamma_k/\Gamma_{k-1} \to N_k \to N_{k-1} \to 1$$

with $\mathbb{Q}$. Note that this central extension is classified by a cohomology class in $H^2(N_{k-1}; \Gamma_k/\Gamma_{k-1})$. We define $N_k^{\mathbb{Q}}$ to be the extension defined by the corresponding class in $H^2(N_{k-1}^{\mathbb{Q}}; \Gamma_k/\Gamma_{k-1} \otimes \mathbb{Q})$. The second assertion follows from construction (exercise). $\blacksquare$

Corollary 3.10. $\pi \to \pi'$ *induces an isomorphism on $H_*(-; \mathbb{Q})$ if and only if it induces an isomorphism $\pi_{\mathbb{Q}} \xrightarrow{\cong} \pi'_{\mathbb{Q}}$.*

Exercise 3.11. Prove or disprove that rationalization is an exact functor.

3.2.2 Rationalization of nilpotent spaces

Definition 3.12 (rational equivalence). $X \xrightarrow{f} Y$ is a *rational homotopy equivalence* if one of the following equivalent conditions holds:

(i) $\pi_*(f) \otimes \mathbb{Q}$ is an isomorphism,

(ii) $H_*(f; \mathbb{Q})$ is an isomorphism.

The equivalence of the two conditions follows by tensoring with $\mathbb{Q}$ from the proof of nilpotent Whitehead theorem.

Definition 3.13 (rational space, rationalization). A nilpotent space X is rational if $H_*(X; \mathbb{Z})$ is rational, or equivalently $\pi_* X$ is rational. A map between nilpotent spaces $X \xrightarrow{f} Y$ is a rationalization if Y is rational and $H_*(f; \mathbb{Z})$ is rationalization, or equivalently $\pi_*(f)$ is a rationalization.

Proposition 3.14 (universal property of rationalization). *Let $X \to X_{\mathbb{Q}}$ be a rationalization. Then any map from X into a rational space $Y_{\mathbb{Q}}$ factors through $X_{\mathbb{Q}}$ uniquely, all up to homotopy.*

$$
\begin{array}{ccc}
X & \longrightarrow & X_{\mathbb{Q}} \\
& \searrow & \downarrow \\
& & Y_{\mathbb{Q}}
\end{array}
$$

Proof. The obstructions for finding such a map, and a homotopy between two such maps, both lie in the cohomology of $X_{\mathbb{Q}}$ with (possibly twisted) coefficients in homotopy groups of $Y_{\mathbb{Q}}$. Since $Y_{\mathbb{Q}}$ is rational, and $X \to X_{\mathbb{Q}}$ induces an isomorphism on rational homology (and therefore cohomology), we see that all obstructions vanish. ∎

Corollary 3.15. *The rationalization, if exists, is unique up to unique homotopy.*

3.3 Construction of rationalization

We present two constructions of rationalization without proof.

3.3.1 Cellular

Let X be CW complex, then X is obtained from attaching cells onto previous skeleton via attaching maps $S^k \to$ previous skeleton. Assume the previous skeleton has been rationalized, then it remains to rationalize the sphere and the attaching map.

To rationalize S^k, recall that $\mathbb{Q} = \varinjlim(\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \mathbb{Z} \to \cdots)$, we similarly define:

$$S^k_{\mathbb{Q}} := \varinjlim(S^k \xrightarrow{2} S^k \xrightarrow{3} S^k \xrightarrow{4} S^k \to \cdots)$$

where $S^k \xrightarrow{n} S^k$ is a degree n map. The attaching map can be rationalized using the universal property.

$$
\begin{array}{ccc}
\vee_\alpha S^k_\alpha & \longrightarrow & \text{previous skeleton} \\
\downarrow & & \downarrow \\
\vee_\alpha (S^k_\alpha)_{\mathbb{Q}} & \dashrightarrow & (\text{previous skeleton})_{\mathbb{Q}}
\end{array}
$$

The rationalization of the new skeleton can be defined to be the cofiber of the dashed arrow. To begin the process, we may assume X has only one zero cell.

3.3.2 Postnikov tower

Similar to the process of constructing the rationalization of a nilpotent group, we can tensor the Postnikov system of a space with $\mathbb{Q}$ inductively. Let $X\langle k \rangle$ be the k-th stage in Postnikov system of X, i.e. $X \to X\langle k \rangle$ is an isomorphism on π_i for $i \leq k$, and $\pi_i X\langle k \rangle = 0$ for $i > k$. There is an inductive tower of fiberations

$$\cdots \to X\langle k \rangle \to X\langle k-1 \rangle \to \cdots \to X\langle 2 \rangle \to X\langle 1 \rangle$$

with inductive fiberation

$$X\langle k\rangle \longleftarrow K(\pi_k X, k)$$
$$\downarrow$$
$$X\langle k-1\rangle$$

that is classified by a single invariant, the (only) obstruction class to finding a section

$$a^{k+1} \in H^{k+1}(X\langle k-1\rangle; \pi_k X).$$

Now suppose $X\langle k-1\rangle$ has be rationalized, then we define $X\langle k\rangle_{\mathbb{Q}}$ to be the total space of the fiberation

$$X\langle k\rangle_{\mathbb{Q}} \longleftarrow K(\pi_k X \otimes \mathbb{Q}, k)$$
$$\downarrow$$
$$X\langle k-1\rangle_{\mathbb{Q}}$$

classified by the corresponding class

$$a_{\mathbb{Q}}^{k+1} \in H^{k+1}(X\langle k-1\rangle_{\mathbb{Q}}; \pi_k X \otimes \mathbb{Q}).$$

The first stage of this process is the rationalization of a nilpotent group.

3.4 Equivalence of homotopy categories

Throughout this section, spaces and DGAs are connected and nilpotent unless otherwise stated. By a space we shall ambiguously mean a topological space, a simplicial complex or a simplicial set; they have equivalent homotopy theories.

Recall that we already have a functor

$$\text{simplicial complex} \to \text{DGA}, \quad X \to \mathscr{A}_X = \mathbb{Q}\text{-polynomial forms on } X$$

and it remains to define a "inverse". For this, recall that for a rational commutative algebra A, (rational points of) its spectrum is identified with $Hom(A, \mathbb{Q})$, where $Hom(-, -)$ means algebra morphisms. In analogy, we define for a rational commutative DGA $\mathscr{A}$, its *spectrum*

$$\text{Spec}\,\mathscr{A} := Hom(\mathscr{A}, \mathscr{A}_{\Delta^\bullet}),$$

where $\mathscr{A}_{\Delta^n}$ is the DGA of rational polynomial forms on Δ^n and $Hom(-, -)$ now means DGA morphisms. Note that since $\Delta^\bullet$ is naturally a co-simplicial set, $\mathscr{A}_{\Delta^\bullet}$ carries the structure of a

simplicial DGA and consequently $\operatorname{Spec}\mathscr{A}$ becomes a simplicial set; after geometric realization we obtain

$$\text{DGA} \to \text{simplicial complex}, \quad \mathscr{A} \to \langle\mathscr{A}\rangle = \text{realization of } \operatorname{Spec}\mathscr{A}.$$

By composing the two functors

$$\text{simplicial complex} \xrightarrow[\langle-\rangle]{\mathbb{Q}\text{-poly. forms}} \text{DGA}$$

we have two endo-functors, each of which carries a canonical natural transformation from identity:

(i) $\mathscr{A} \xrightarrow{\pi} \mathbb{Q}$-polynomial forms on $\langle\mathscr{A}\rangle$, defined by evaluation:

$$\mathscr{A} \ni a \to \{a_\sigma = \sigma(a)\}.$$

(ii) $X \xrightarrow{\rho} \langle\mathscr{A}_X\rangle$, defined by projection:

$$X \supset \sigma \to \langle\sigma\rangle : \omega \mapsto \omega_\sigma.$$

We wish to show that (i) is a quasi-isomorphism and (ii) is rationalization. They both follow from

Theorem 3.16 (adjunction). *The functors*

$$\textit{simplicial complex} \xrightarrow[\langle-\rangle]{\mathbb{Q}\text{-poly. forms}} \textit{DGA}$$

are "homotopy adjoints". For X a simplicial complex and $\mathscr{A}$ a DGA, we have "homotopy adjunction":

$$\textit{Hom}(X, \langle\mathscr{A}\rangle) \textit{ mod. homotopy} \sim \textit{Hom}(\mathscr{A}, \mathscr{A}_X) \textit{ mod. homotopy}.$$

Chapter 4

Applications

4.1 Formality of Kähler manifolds

4.1.1 Characterization of formality

Recall that a DGA if formal if it is quasi-isomorphic (as DGA) to its cohomology; the minimal model of such a DGA is therefore a formal consequence of its cohomology ring.

Theorem 4.1. *A nilpotent minimal algebra $\mathscr{A}$ is formal iff the grading automorphisms of $H\mathscr{A}$ lift to $\mathscr{A}$, or iff all automorphisms of $H\mathscr{A}$ lift to $\mathscr{A}$.*

By grading automorphism, we mean the action of the multiplicative group corresponding to the grading; on H^i the multiplicative group acts by weight i, i.e. α acts via multiplication by α^i.

Example 4.2. Let $\mathscr{A} = \wedge(x_1, \ldots, x_n, y_1, y_2)$, $|x_i| = 2$, $dx_i = 0$, $dy_1 = x_1^2 + \cdots + x_n^2$, $dy_2 = x_1^3 + \cdots + x_n^3$. If α is a lift of grading automorphism on $H\mathscr{A}$, then α must act on x_i by weight 2 (since it does so on x_i in cohomology) and the differential $dy_1 = x_1^2 + \cdots + x_n^2$ forces σ to act on y by weight 4; similarly it acts on y_2 by weight 6. This determines the action of α, for instance it acts on $y_1 y_2$ by weight $4 + 6 = 10$; note $|y_1 y_2| = 3 + 5 = 8$.

Proof. If $\mathscr{A}$ is formal, we have by definition a dga quasi-isomorphism $\mathscr{A} \to H\mathscr{A}$ which must be surjective, then obstruction theory (Proposition 2.15) implies that all automorphisms lift.

Now suppose the grading automorphism of $H\mathscr{A}$ lift to $\mathscr{A}$, we obtain a decomposition (via diagonalizable part of the action) of $\mathscr{A}$ into its weight subspaces. We can prove by induction (using the short exact sequences in the proof of Proposition 2.33) that weight $\geq$ degree, i.e. the weights carried by degree k subspace $\mathscr{A}^k$ are $\geq k$. The weights on new generators are determined

by structural equation

$$dx = \text{products of previous generators.}$$

Since both weight and degree are additive with respect to product, the right hand side has degree $k+1$ and therefore has weight $\geq k+1$ by induction hypothesis. This forces new generators to have weights $\geq k+1$, except for those satisfying $dx = 0$. Those generators in $\ker d$ (spherical cohomology) would have weight k as they pass to cohomology. The same argument also proves that $\mathscr{A}^1$ carries weight ≥ 1.

Consider now $S := \{\text{weight} = \text{degree}\}$ and $I := \{\text{weight} > \text{degree}\}$. It is clear that (i) $S \subset \ker d$ as d increases degree but preserves weight, (ii) S is a subalgebra and I is a multiplicative ideal as both degree and weight are additive with respect to product. Then $\mathscr{A} = S \oplus I$ implies that $H\mathscr{A} = S/S \cap dI + \text{extra from } I$, but since all cohomology classes can be represented by cocyles in S (the weights are preserved by d and therefore weight subspaces are preserved by d, so the weight decomposition passes to cohomology but cohomology has weight equal to degree) we see $H\mathscr{A} = S/S \cap dI$. Therefore the quotient map of DGA's

$$\mathscr{A} \to \mathscr{A}/(I,dI) = S/S \cap dI$$

is a quasi-isomorphism between $\mathscr{A}$ and $H\mathscr{A}$. ∎

Example 4.3 (non-formal, Massey product)**.** Let $\mathscr{A} = \wedge(x,y,z)$, $|x| = |y| = 3$, $|z| = 5$, $dx = dy = 0$, $dz = xy$. Then $H\mathscr{A}$ is spanned by $x,y; xz, yz; xyz$. Now suppose the grading automorphism of $H\mathscr{A}$ lifts to $\mathscr{A}$, then x,y must have weight 3 and consequently z has weight 6. This forces xz, yz to have weight 9 and xyz to have weight 12; but they respectively have weights 8 and 11 in cohomology. This means $\mathscr{A}$ is not formal. Note that in this example xz and yz in cohomology are the "primary obstructions" to formality; they are Massey triple products $< x,x,y >= xz$ and $< y,y,z >= yz$.

Proposition 4.4. *If $\mathscr{A}$ is formal, then all Massey (triple and higher order) products vanish.*

Proof. Massey products have wrong weights. ∎

Remark 4.5. The vanishing of all Massey products will <u>not</u> automatically imply formality; they need to vanish in a "coherent way".

Corollary 4.6 (Formality is independent of field extension)**.** *Let $k \subset K$ be a field extension (of characteristic zero fields) and $\mathscr{A}$ a nilpotent minimal DGA with finite dimensional cohomology over k. Then $\mathscr{A}$ is formal over k iff $\mathscr{A} \otimes_k K$ is formal over K.*

Proof. The automorphism group G of $\mathscr{A}$ and G' of $H\mathscr{A}$ are k-algebraic groups, equipped with a natural k-morphism $\rho : G \to G'$. Consider now x a k-point of G', then $\rho^{-1}(x)$ is a principal homogeneous space (i.e. torsor) of the group $\rho^{-1}(e)$ (all defined over k); note that $\rho^{-1}(x)$ has a K-point since $\mathscr{A}$ is formal over K. The obstruction to finding a k-point of $\rho^{-1}(x)$ now lies in

$$H^1(\mathrm{Gal}(K/k), \rho^{-1}(e)).$$

Since $\rho^{-1}(e)$ is unipotent (Proposition 2.33), this cohomology group vanishes. ∎

4.1.2 Principle of two types

Theorem 4.7 (Deligne-Griffiths-Morgan-Sullivan). *Compact Kähler manifolds are formal over $\mathbb{R}$ and therefore formal over $\mathbb{Q}$.*

Proof. The de Rham algebra of complex-valued differential forms on a Kähler manifolds carries, via its Hodge structure, an action by Deligne torus ($\mathbb{C}^*$ viewed as a real algebraic group) in which the multiplicative group $\mathbb{R}^*$ embeds. This action descends to cohomology so that the corresponding action of the multiplicative group has weight equal to degree. ∎

The same argument proves that any complex manifolds whose Frölicher (Hodge-to-de Rham) spectral sequence degenerates on E_1-page is formal. This is true exactly when the manifold satisfies the so-called dd^c-lemma (or $\partial\overline{\partial}$-lemma). We now give another proof of formality based directly on dd^c-lemma.

Proof of Theorem 4.7. Consider on a complex manifold the diagram of DGAs:

$$
\begin{array}{ccc}
(d^c\text{-closed forms}, d) & \overset{i}{\longrightarrow} & (\text{all forms}, d) \\
\downarrow{\scriptstyle p} & & \\
\text{cohomology} & &
\end{array}
$$

where i is the canonical inclusion and p the projection onto d^c-cohomology. On a compact Kähler manifold (or a dd^c-manifold), Hodge theory (d^c-harmonic forms represent cohomology) implies that both p and i are quasi-isomorphisms. Therefore by obstruction theory the minimal model of all forms maps to cohomology inducing an isomorphism on cohomology; in particular the minimal models of all forms and cohomology coincide. ∎

Exercise 4.8. Compute the minimal model of a K3-surface. This is significantly easier over $\mathbb{C}$ than over $\mathbb{Q}$.

Example 4.9. Let G be a connected Lie group and $T \subset G$ a maximal torus. Then G/T is formal. Indeed let $K \subset G$ be its maximal compact subgroup, then G retracts to K and G/T retracts to K/T; the latter is a Kähler manifold.

Exercise 4.10 (Alternative proof of that Lie groups are formal). Deduce from the example above that connected Lie groups are formal. Hint: using the fiberation $T \to G \to G/T$, a model for G is built from a model for G/T by adding new generators $x_1, \ldots, x_r$ of degree one corresponding to generators of cohomology of T with differential hitting (representatives of) Euler classes.

These ideas, by usage of Deligne's mixed Hodge theory, can be extended to open smooth complex algebraic varieties.

Theorem 4.11 (Morgan). *Let $V \subset \overline{V}$ be the complement of a divisor D with normal crossings in a non-singular compact variety[1].*

(i) *Rational minimal model of V carries a mixed Hodge structure, unique up to inner automorphism.*

(ii) *Let $D^p \subset D$ be the subvariety of D with multiplicity p, define $\widetilde{D}^p$ to be the normalization of D^p, $p \geq 1$ and to be $\overline{V}$, $p = 0$. Then the rational homotopy type of V is determined by:*

  (a) *the cohomology groups of the $\widetilde{D}^p$, $p \geq 0$;*
  (b) *the multiplication map $H(\widetilde{D}^p) \otimes H(\widetilde{D}^q) \to H(\widetilde{D}^{p+q})$ for $p, q \geq 0$; and*
  (c) *the Gysin maps $H(\widetilde{D}^p) \to H(\widetilde{D}^{p-1})$.*

4.2 Automorphism of integral homotopy type

The automorphism of an integral homotopy type is an arithmetic group.

4.3 Classification of manifolds

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[1]This covers all open smooth complex algebraic varieties by Hironaka's theorem.

Appendix A

Algebraic and Arithmetic groups

We here collects some facts about algebraic matrix groups and arithmetic groups.

A.1   Levi decomposition

A.2   Chevalley's trick

A.3   Principal homogeneous space

A.4   Galois cohomology

Theorem A.1. *Let G be a unipotent group defined over a characteristic zero field k. Then*

$$H^1(k;G) = 0.$$

A.5   Borel-Serre finiteness